Generalized Nash Equilibrium Problem: existence, uniqueness and reformulations

Didier Aussel

Univ. de Perpignan, France

CIMPA-UNESCO school, Delhi
November 25 - December 6, 2013
Outline of the 7 lectures

- **Generalized Nash Equilibrium**
  - a- Reformulations
  - b- Existence of equilibrium

- **Variational inequalities: motivations, definitions**
  - a- Motivations, definitions
  - b- Existence of solutions
  - c- Stability
  - d- Uniqueness

- **Quasiconvex optimization**
  - a- Classical subdifferential approach
  - b- Normal approach

- **Whenever a set-valued map is not really set-valued...**

- **Quasivariational inequalities**
Outline of lecture 2

I- Classical approaches of quasiconvex analysis

II- Normal approach
   a- First definitions
   b- Adjusted sublevel sets and normal operator

III- Quasiconvex optimization
   a- Optimality conditions
   b- Convex constraint case
   c- Nonconvex constraint case
I - Introduction to quasiconvex optimization
A function \( f : X \to \mathbb{R} \cup \{+\infty\} \) is said to be \textit{quasiconvex} on \( K \) if,

\[
\text{for all } x, y \in K \text{ and all } t \in [0, 1], \\
\quad f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.
\]

or

\[
\text{for all } \lambda \in \mathbb{R}, \text{ the sublevel set } \\
\quad S_\lambda = \{x \in X : f(x) \leq \lambda\} \text{ is convex.}
\]
A function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be \textit{quasiconvex} on $K$ if,

\textit{for all $x, y \in K$ and all $t \in [0, 1]$},

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$ 

\textit{or}

\textit{for all $\lambda \in \mathbb{R}$, the sublevel set}

$$S_\lambda = \{x \in X : f(x) \leq \lambda\} \text{ is convex.}$$

\textit{$f$ differentiable}

$f$ is quasiconvex iff $df$ is quasimonotone

\textit{iff $df(x)(y - x) > 0 \Rightarrow df(y)(y - x) \geq 0$}
A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be quasiconvex on $K$ if,

for all $x, y \in K$ and all $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$ 

or

for all $\lambda \in \mathbb{R}$, the sublevel set

$$S_\lambda = \{x \in X : f(x) \leq \lambda\}$$

is convex.

- $f$ differentiable
  - $f$ is quasiconvex iff $df$ is quasimonotone
    if $df(x)(y - x) > 0 \Rightarrow df(y)(y - x) \geq 0$
  - $f$ is quasiconvex iff $\partial f$ is quasimonotone
    if $\exists x^* \in \partial f(x) : \langle x^*, y - x \rangle > 0$
    $\Rightarrow \forall y^* \in \partial f(y), \langle y^*, y - x \rangle \geq 0$
A function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on $K$ if,

for all $x, y \in K$ and all $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$ 

or

for all $\lambda \in \mathbb{R}$, the sublevel set

$$S_\lambda = \{x \in X : f(x) \leq \lambda\}$$ is convex.

A function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *semistrictly quasiconvex* on $K$ if, $f$ is quasiconvex and for any $x, y \in K$,

$$f(x) < f(y) \Rightarrow f(z) < f(y), \quad \forall z \in [x, y].$$

convex $\Rightarrow$ semistrictly quasiconvex $\Rightarrow$ quasiconvex
Why not a subdifferential for quasiconvex programming?
Why not a subdifferential for quasiconvex programming?

- No (upper) semicontinuity of $\partial f$ if $f$ is not supposed to be Lipschitz
Why not a subdifferential for quasiconvex programming?

- No (upper) semicontinuity of $\partial f$ if $f$ is not supposed to be Lipschitz

- No sufficient optimality condition

\[ \bar{x} \in S_{str}(\partial f, C) \not\Rightarrow \bar{x} \in \arg \min_C f \]
II - Normal approach

a- First definitions
A first approach

Sublevel set:

\[ S_\lambda = \{ x \in X : f(x) \leq \lambda \} \]
\[ S_{\lambda}^\supset = \{ x \in X : f(x) < \lambda \} \]

Normal operator:

Define \( N_f(x) : X \to 2^{X^*} \) by

\[
N_f(x) = N(S_{f(x)}, x) = \{ x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \ \forall y \in S_{f(x)} \}.
\]

With the corresponding definition for \( N_f^\supset(x) \)
$N_f(x) = N(S_f(x), x)$ has no upper-semicontinuity properties

$N^>_f(x) = N(S^>_f(x), x)$ has no quasimonotonicity properties

Example

Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

\[
f(a, b) = \begin{cases} 
|a| + |b|, & \text{if } |a| + |b| \leq 1 \\
1, & \text{if } |a| + |b| > 1
\end{cases}
\]

Then $f$ is quasiconvex.

Consider $x = (10, 0), x^* = (1, 2), y = (0, 10)$ and $y^* = (2, 1)$. We see that $x^* \in N^<(x)$ and $y^* \in N^<(y)$ (since $|a| + |b| < 1$ implies $(1, 2) \cdot (a - 10, b) \leq 0$ and $(2, 1) \cdot (a, b - 10) \leq 0$) while $\langle x^*, y - x \rangle > 0$ and $\langle y^*, y - x \rangle < 0$. Hence $N^<$ is not quasimonotone.
• $N_f(x) = N(S_f(x), x)$ has no upper-semicontinuity properties
• $N^>_f(x) = N(S^>_f(x), x)$ has no quasimonotonicity properties

Example
Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(a, b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \leq 1 \\ 1, & \text{if } |a| + |b| > 1 \end{cases}$$

Then $f$ is quasiconvex.
But ...

- \( N_f(x) = N(S_f(x), x) \) has no upper-semicontinuity properties
- \( N_f^>(x) = N(S_f^>(x), x) \) has no quasimonotonicity properties

**Example**

Define \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) by

\[
    f(a, b) = \begin{cases} 
        |a| + |b|, & \text{if } |a| + |b| \leq 1 \\
        1, & \text{if } |a| + |b| > 1
    \end{cases}
\]

Then \( f \) is quasiconvex.

Consider \( x = (10, 0), x^* = (1, 2), y = (0, 10) \) and \( y^* = (2, 1) \).

We see that \( x^* \in N^<(x) \) and \( y^* \in N^<(y) \) (since \( |a| + |b| < 1 \) implies \( (1, 2) \cdot (a - 10, b) \leq 0 \) and \( (2, 1) \cdot (a, b - 10) \leq 0 \) while \( \langle x^*, y - x \rangle > 0 \) and \( \langle y^*, y - x \rangle < 0 \). Hence \( N^< \) is not quasimonotone.
But ...another example

- $N_f(x) = N(S_{f(x)}, x)$ has no upper-semicontinuity properties
- $N_f^>(x) = N(S_{f(x)}^>, x)$ has no quasimonotonicity properties

Example

![Graphical representation](image)

Then $f$ is quasiconvex.
But ...another example

- $N_f(x) = N(S_f(x), x)$ has no upper-semicontinuity properties
- $N_f^>(x) = N(S_f^>(x), x)$ has no quasimonotonicity properties

Example

Then $f$ is quasiconvex.

We easily see that $N(x)$ is not upper semicontinuous....
But ...another example

- $N_f(x) = N(S_f(x), x)$ has no upper-semicontinuity properties
- $N_f^>(x) = N(S_f^>(x), x)$ has no quasimonotonicity properties

Example

Then $f$ is quasiconvex.

We easily see that $N(x)$ is not upper semicontinuous....

These two operators are essentially adapted to the class of semi-strictly quasiconvex functions. Indeed in this case, for each $x \in \text{dom } f \setminus \arg \min f$, the sets $S_f(x)$ and $S_f^<(x)$ have the same closure and $N_f(x) = N_f^<(x)$. 

Didier Aussel  
Generalized Nash Equilibrium Problem: existence, uniqueness and
II - Normal approach

b- Adjusted sublevel sets and normal operator
**Definition**

**Adjusted sublevel set**

For any \( x \in \text{dom} \, f \), we define

\[
S^a_f(x) = S_f(x) \cap \overline{B}(S^{<}_f(x), \rho_x)
\]

where \( \rho_x = \text{dist}(x, S^{<}_f(x)) \), if \( S^{<}_f(x) \neq \emptyset \)

and \( S^a_f(x) = S_f(x) \) if \( S^{<}_f(x) = \emptyset \).
**Definition**

**Adjusted sublevel set**

For any $x \in \text{dom } f$, we define

$$S_a^f(x) = S_f(x) \cap \overline{B}(S_f^{<}(x), \rho_x)$$

where $\rho_x = \text{dist}(x, S_f^{<}(x))$, if $S_f^{<}(x) \neq \emptyset$

and $S_a^f(x) = S_f(x)$ if $S_f^{<}(x) = \emptyset$.

- $S_a^f(x)$ coincides with $S_f(x)$ if $\text{cl}(S_f^{>}(x)) = S_f(x)$

---

*Didier Aussel*

Generalized Nash Equilibrium Problem: existence, uniqueness and
**Definition**

**Adjusted sublevel set**

For any $x \in \text{dom } f$, we define

$$S^a_f(x) = S_f(x) \cap \overline{B}(S^<_f(x), \rho_x)$$

where $\rho_x = \text{dist}(x, S^<_f(x))$, if $S^<_f(x) \neq \emptyset$

and $S^a_f(x) = S_f(x)$ if $S^<_f(x) = \emptyset$.

- $S^a_f(x)$ coincides with $S_f(x)$ if $\text{cl}(S^>_f(x)) = S_f(x)$

  e.g. $f$ is semistrictly quasiconvex
**Adjusted sublevel set**

For any $x \in \text{dom } f$, we define

$$S_f^a(x) = S_f(x) \cap \overline{B}(S^<_f(x), \rho_x)$$

where $\rho_x = \text{dist}(x, S^<_f(x))$, if $S^<_f(x) \neq \emptyset$

and $S_f^a(x) = S_f(x)$ if $S^<_f(x) = \emptyset$.

- $S_f^a(x)$ coincides with $S_f(x)$ if $\text{cl}(S^>_f(x)) = S_f(x)$

  e.g. $f$ is semistrictly quasiconvex

**Proposition**

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function, with domain $\text{dom } f$. Then

$$f \text{ is quasiconvex } \iff S_f^a(x) \text{ is convex }, \forall x \in \text{dom } f.$$
Let us suppose that $S_f^2 (u)$ is convex for every $u \in \text{dom } f$. We will show that for any $x \in \text{dom } f$, $S_f(x)$ is convex.

If $x \in \arg \min f$ then $S_f(x) = S_f^a(x)$ is convex by assumption.

Assume now that $x \notin \arg \min f$ and take $y, z \in S_f(x)$.

If both $y$ and $z$ belong to $B \left( S_f^< (x), \rho_x \right)$, then $y, z \in S_f^a(x)$ thus $[y, z] \subseteq S_f^a(x) \subseteq S_f(x)$.

If both $y$ and $z$ do not belong to $B \left( S_f^< (x), \rho_x \right)$, then

$$f(x) = f(y) = f(z), \quad \bar{S}_f^<(z) = \bar{S}_f^<(y) = \bar{S}_f^<(x)$$

and $\rho_y, \rho_z$ are positive. If, say, $\rho_y \geq \rho_z$ then $y, z \in B \left( \bar{S}_f^<(y), \rho_y \right)$ thus

$$y, z \in S_f^a(y) \quad \text{and} \quad [y, z] \subseteq S_f^a(y) \subseteq S_f(y) = S_f(x).$$
Finally, suppose that only one of $y$, $z$, say $z$, belongs to $\overline{B}(S_{f(x)}^\prec, \rho_x)$ while $y \notin \overline{B}(S_{f(x)}^\prec, \rho_x)$. Then

$$f(x) = f(y), \quad S_{f(y)}^\prec = S_{f(x)}^\prec \quad \text{and} \quad \rho_y > \rho_x$$

so we have $z \in \overline{B} \left( S_{f(x)}^\prec, \rho_x \right) \subseteq \overline{B} \left( S_{f(y)}^\prec, \rho_y \right)$ and we deduce as before that $[y, z] \subseteq S_f^a(y) \subseteq S_f(y) = S_f(x)$.

The other implication is straightforward.\[\square\]
Adjusted sublevel set:
For any $x \in \text{dom} \ f$, we define

$$S^a_f(x) = S_f(x) \cap \overline{B}(S^<_f(x), \rho_x)$$

where $\rho_x = \text{dist}(x, S^<_f(x))$, if $S^<_f(x) \neq \emptyset$.

Adjusted normal operator:

$$N^a_f(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \ \forall y \in S^a_f(x)\}$$
Example

Generalized Nash Equilibrium Problem: existence, uniqueness and
\begin{align*}
\overline{B}(S_{f(x)}^<, \rho_x) \\
S_f^a(x) &= S_f(x) \cap \overline{B}(S_{f(x)}^<, \rho_x)
\end{align*}
Example

\[ S_f^a(x) = S_f(x) \cap \bar{B}(S_f^<(x), \rho_x) \]

\[ N_f^a(x) = \{ x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \ \forall \ y \in S_f^a(x) \} \]
Let us draw the normal operator value $N_f^a(x, y)$ at the points $(x, y) = (0.5, 0.5), (x, y) = (0, 1), (x, y) = (1, 0), (x, y) = (1, 2), (x, y) = (1.5, 0)$ and $(x, y) = (0.5, 2)$. 
Let us draw the normal operator value $N_{f}^{a}(x, y)$ at the points 
$(x, y) = (0.5, 0.5), (x, y) = (0, 1), (x, y) = (1, 0), (x, y) = (1, 2), \ (x, y) = (1.5, 0)$ and $(x, y) = (0.5, 2)$.

Operator $N_{f}^{a}$ provide information at any point!!!
One can have

\[ N_f^a(x) \nsubseteq \text{cone}(\partial f(x)) \quad \text{or} \quad \text{cone}(\partial f(x)) \nsubseteq N_f^a(x) \]

**Proposition**

\[ f \text{ is quasiconvex and } x \in \text{dom } f \]

If there exists \( \delta > 0 \) such that \( 0 \notin \partial L f(B(x, \delta)) \) then

\[ \left[ \text{cone}(\partial L f(x)) \cup \partial \infty f(x) \right] \subset N_f^a(x). \]

**Proposition**

\[ f \text{ is lsc semistrictly quasiconvex and } 0 \notin \partial f(X) \text{ then} \]

\[ \text{cone}(\partial f(x)) \subset N_f^a(x). \]

If additionally \( \partial = \partial^{\uparrow} \) and \( f \) is Lipschitz then

\[ N_f^a(x) = \text{cone}(\partial f(x)). \]
Basic properties of $N^a_f$

Nonemptyness:

**Proposition**

Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be lsc. Assume that rad. continuous on $\text{dom } f$ or $\text{dom } f$ is convex and $\text{int}S_{\lambda} \neq \emptyset$, $\forall \lambda > \inf_X f$. Then

$f$ is quasiconvex $\iff N^a_f(x) \setminus \{0\} \neq \emptyset, \ \forall x \in \text{dom } f \setminus \text{arg min } f$.

Quasimonotonicity:

The normal operator $N^a_f$ is always quasimonotone.
Upper sign-continuity

- $T : X \to 2^{X^*}$ is said to be upper sign-continuous on $K$ iff for any $x, y \in K$, one have:

$$\forall t \in ]0, 1[, \inf_{x^* \in T(x_t)} \langle x^*, y - x \rangle \geq 0$$

$$\Rightarrow \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0$$

where $x_t = (1 - t)x + ty$.

upper semi-continuous

$\Downarrow$

upper hemicontinuous

$\Downarrow$

upper sign-continuous
Definition

Let $T : K \rightarrow 2^{X^*}$ be a set-valued map.

$T$ is called locally upper sign-continuous on $K$ if, for any $x \in K$ there exist a neigh. $V_x$ of $x$ and a upper sign-continuous set-valued map $\Phi_x(\cdot) : V_x \rightarrow 2^{X^*}$ with nonempty convex $w^*$-compact values such that $\Phi_x(y) \subseteq T(y) \setminus \{0\}, \forall y \in V_x$
Definition

Let \( T : K \rightarrow 2^{\mathbb{X}^*} \) be a set-valued map.

\( T \) is called locally upper sign-continuous on \( K \) if, for any \( x \in K \) there exist a neigh. \( V_x \) of \( x \) and a upper sign-continuous set-valued map \( \Phi_x(\cdot) : V_x \rightarrow 2^{\mathbb{X}^*} \) with nonempty convex \( w^* \)-compact values such that
\[
\Phi_x(y) \subseteq T(y) \setminus \{0\}, \quad \forall y \in V_x
\]

Continuity of normal operator

Proposition

Let \( f \) be lsc quasiconvex function such that \( \text{int}(S_\lambda) \neq \emptyset, \quad \forall \lambda > \inf f. \)

Then \( N_f \) is locally upper sign-continuous on \( \text{dom} \ f \setminus \text{arg min} \ f. \)
Proposition

If $f$ is quasiconvex such that $\text{int}(S_\lambda) \neq \emptyset$, $\forall \lambda > \inf f$ and $f$ is lsc at $x \in \text{dom } f \setminus \text{arg min } f$,

Then $N_f^a$ is norm-to-$w^*$ cone-usc at $x$.

A multivalued map with conical valued $T : X \rightarrow 2^{X^*}$ is said to be cone-usc at $x \in \text{dom } T$ if there exists a neighbourhood $U$ of $x$ and a base $C(u)$ of $T(u)$, $u \in U$, such that $u \rightarrow C(u)$ is usc at $x$. 
Let $f : X \rightarrow \mathbb{IR} \cup \{+\infty\}$ quasiconvex

**Question:** Is it possible to characterize the functions $g : X \rightarrow \mathbb{IR} \cup \{+\infty\}$ quasiconvex such that $N_f^a = N_g^a$?
Let \( f : X \rightarrow \mathbb{R} \cup \{+\infty\} \) quasiconvex

**Question:** Is it possible to characterize the functions \( g : X \rightarrow \mathbb{R} \cup \{+\infty\} \) quasiconvex such that \( N^a_f = N^a_g \)?

**A first answer:**
Let \( C = \{g : X \rightarrow \mathbb{R} \cup \{+\infty\} \text{ cont. semistrictly quasiconvex such that argmin } f \text{ is included in a closed hyperplane}\} \)

Then, for any \( f, g \in C \),

\[
N^a_f = N^a_g \iff g \text{ is } N^a_f \setminus \{0\}-\text{pseudoconvex} \\
\iff \exists x^* \in N^a_f(x) \setminus \{0\} : \langle x^*, y - x \rangle \geq 0 \Rightarrow g(x) \leq g(y).
\]
Integration of $N_f^a$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex

**Question:** Is it possible to characterize the functions $g : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex such that $N_f^a = N_g^a$?

**A first answer:**
Let $C = \{g : X \to \mathbb{R} \cup \{+\infty\} \text{ cont. semistrictly quasiconvex such that } \arg\min f \text{ is included in a closed hyperplane}\}$

Then, for any $f, g \in C$,

$$N_f^a = N_g^a \iff g \text{ is } N_f^a \setminus \{0\}-\text{pseudoconvex} \iff \exists x^* \in N_f^a(x) \setminus \{0\} : \langle x^*, y - x \rangle \geq 0 \Rightarrow g(x) \leq g(y).$$

**General case:** open question
IV

Quasiconvex programming

a- Optimality conditions
Quasiconvex programming

Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $K \subseteq \text{dom } f$ be a convex subset.

\[(P) \quad \text{find } \bar{x} \in K : f(\bar{x}) = \inf_{x \in K} f(x)\]
Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $K \subseteq \text{dom } f$ be a convex subset.

$$(P) \quad \text{find } \bar{x} \in K : f(\bar{x}) = \inf_{x \in K} f(x)$$

**Perfect case: $f$ convex**

$f : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex function

$K$ a nonempty convex subset of $X$, $\bar{x} \in K + C.Q.$

Then

$$f(\bar{x}) = \inf_{x \in K} f(x) \iff \bar{x} \in S_{str}(\partial f, K)$$
Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $K \subseteq \text{dom } f$ be a convex subset.

\[(P) \quad \text{find } \bar{x} \in K : f(\bar{x}) = \inf_{x \in K} f(x)\]

**Perfect case:** $f$ convex

$f : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex function

$K$ a nonempty convex subset of $X$, $\bar{x} \in K$ + C.Q.

Then

$f(\bar{x}) = \inf_{x \in K} f(x) \iff \bar{x} \in S_{str}(\partial f, K)$

**What about $f$ quasiconvex case?**

$\bar{x} \in S_{str}(\partial f(\bar{x}), K) \Rightarrow \bar{x} \in \arg \min_{K} f$
Theorem

\[ f : X \to \mathbb{R} \cup \{+\infty\} \text{ quasiconvex, radially cont. on } \text{dom } f \]

\[ C \subseteq X \text{ such that conv}(C) \subset \text{dom } f. \]

Suppose that \( C \subset \text{int}(\text{dom } f) \) or \( \text{Aff}C = X \).

Then \( \bar{x} \in S(N^a_f \setminus \{0\}, C) \implies \forall x \in C, f(\bar{x}) \leq f(x). \)

where \( \bar{x} \in S(N^a_f \setminus \{0\}, K) \) means that there exists \( \bar{x}^* \in N^a_f(\bar{x}) \setminus \{0\} \) such that

\[ \langle \bar{x}^*, c - x \rangle \geq 0, \quad \forall c \in C. \]
Lemma

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function, radially continuous on $\text{dom } f$. Then $f$ is $N_f^a \setminus \{0\}$-pseudoconvex on $\text{int}(\text{dom } f)$, that is,

$$\exists x^* \in N_f^a(x) \setminus \{0\} : \langle x^*, y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x).$$

Proof.

Let $x, y \in \text{int}(\text{dom } f)$. According to the quasiconvexity of $f$, $N_f^a(x) \setminus \{0\}$ is nonempty. Let us suppose that $\langle x^*, y - x \rangle \geq 0$ with $x^* \in N_f^a(x) \setminus \{0\}$. Let $d \in X$ be such that $\langle x^*, y_n - x \rangle > 0$ for any $n$, where $y_n = y + \frac{1}{n}d$ ($\in \text{dom } f$ for $n$ large enough).

This implies that $y_n \not\in S_f^\prec(x)$ since $x^* \in N_f^a(x) \subset N_f^\prec(x)$.

It follows by the radial continuity of $f$ that $f(y) \geq f(x)$.
Necessary and Sufficient conditions

Proposition

Let $C$ be a closed convex subset of $X$, $\bar{x} \in C$ and $f : X \to \mathbb{R}$ be continuous semistrictly quasiconvex such that $\text{int}(S_{f,c}^{a}(\bar{x})) \neq \emptyset$ and $f(\bar{x}) > \inf_{X} f$.

Then the following assertions are equivalent:

i) $f(\bar{x}) = \min_{C} f$

ii) $\bar{x} \in S_{str}(N_{f}^{a} \setminus \{0\}, C)$

iii) $0 \in N_{f}^{a}(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$. 
Proposition

Let $C$ be a closed convex subset of an Asplund space $X$ and $f : X \to \mathbb{R}$ be a continuous quasiconvex function. Suppose that either $f$ is sequentially normally sub-compact or $C$ is sequentially normally compact. If $\bar{x} \in C$ is such that $0 \not\in \partial L f(\bar{x})$ and $0 \not\in \partial^{\infty} f(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$

then the following assertions are equivalent:

i) $f(\bar{x}) = \min_C f$

ii) $\bar{x} \in \partial^L f(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$

iii) $0 \in N^a_f(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$


Appendix 5 - Normally compactness

A subset $C$ is said to be *sequentially normally compact* at $x \in C$ if for any sequence $(x_k)_k \subset C$ converging to $x$ and any sequence $(x^*_k)_k$, $x^*_k \in N^F(C, x_k)$ weakly converging to 0, one has $\|x^*_k\| \to 0$.

**Examples:**

- $X$ finite dimensional space
- $C$ epi-Lipschitz at $x$
- $C$ convex with nonempty interior

A function $f$ is said to be *sequentially normally subcompact* at $x \in C$ if the sublevel set $S_f(x)$ is sequentially normally compact at $x$.

**Examples:**

- $X$ finite dimensional space
- $f$ is locally Lipschitz around $x$ and $0 \not\in \partial^L f(x)$
- $f$ is quasiconvex with $\text{int}(S_f(x)) \neq \emptyset$
Let $C$ be any nonempty subset of $X$ and $x \in \bar{C}$. The \textit{limiting normal cone} to $C$ at $x$, denoted by $N^L(C, x)$ is defined by

$$N^L(C, x) = \limsup_{x' \to x} N^F(C, x')$$

where the Frèchet normal cone $N^F(C, x')$ is defined by

$$N^F(C, x') = \left\{ x^* \in X^* : \limsup_{u \to x', u \in C} \frac{\langle x^*, u - x' \rangle}{\| u - x' \|} \leq 0 \right\}.$$

The limiting normal cone is closed (but in general non convex)
One can define the *Limiting subdifferential* (in the sense of Mordukhovich), and its asymptotic associated form, of a function \( f : X \to \mathbb{R} \cup \{+\infty\} \) by

\[
\partial^L f(x) = \text{Limsup}_{y \to x} \partial^F f(y)
= \{ x^* \in X^*: (x^*, -1) \in N^L(\text{epi} f, (x, f(x))) \}
\]

\[
\partial^\infty f(x) = \text{Limsup}_{y, \lambda \to x} \lambda \partial^F f(y)
= \{ x^* \in X^*: (x^*, 0) \in N^L(\text{epi} f, (x, f(x))) \}.
\]